

ALGEBRAIC CURVES EXERCISE SHEET 4

Unless otherwise specified, k is an algebraically closed field.

Exercise 4.1. Show that all local rings of the affine line \mathbb{A}_k^1 are isomorphic to the same ring R .

Exercise 4.2. An *affine algebraic group* is an affine variety G , whose underlying set is a group, such that the morphisms $i : G \rightarrow G$, $g \mapsto g^{-1}$ and $m : G \times G \rightarrow G$, $(g, h) \mapsto gh$ are polynomial maps. Let $V_1 = \mathbb{A}_k^1 - \{0\}$ and $V_2 = V(xy - 1)$. From the first exercise, we call R the local ring of \mathbb{A}_k^1 at any point.

- (1) Show that $\mathcal{O}(V_1) = k[x, x^{-1}] = k[x, y]/(xy - 1)$.
- (2) Construct a morphism $V_2 \rightarrow \mathbb{A}_k^1$ whose image is V_1 .
- (3) Show that the local ring of V_2 at any point is isomorphic to R . Are V_2 and \mathbb{A}_k^1 isomorphic?
- (4) Show that V_2 can be endowed with a structure of affine algebraic group.

Exercise 4.3. Let $V = V(y^2 - x^3)$. Let $\varphi : \mathbb{A}_k^1 \rightarrow V$ be the morphism defined by $\varphi(t) = (t^2, t^3)$. From the first exercise, we call R the local ring of \mathbb{A}_k^1 at any point.

- (1) Show φ is a bijective morphism, but is not an isomorphism.
- (2) Let $P \in V$. Is the local ring of V at P isomorphic to R ?

Exercise 4.4. Let $V = V(Y^2 - X^2(X + 1))$ and x, y the residues of X, Y in $\Gamma(V)$. Let $z = \frac{y}{x} \in k(V)$. Find the poles of z and z^2 .

Exercise 4.5.

- (1) Prove Corollary 2.9 from class: Let V be a quasi-affine variety and $f, g \in \mathcal{O}(V)$ two regular functions, such that $f|_U = g|_U$ for some non-empty open $U \subset V$. Then $f = g$.
- (2) Let V be an affine variety and $f \in k(V)$ a rational function. Show that f defines a continuous function $U \rightarrow k$, for some non empty open subset $U \subset V$. Furthermore f is uniquely determined by this function.

Exercise 4.6. * Let $F \in k[x, y]$ be an irreducible polynomial of degree at most 2. Show that $V(F)$ is either isomorphic to $V_1 = \mathbb{A}_k^1$ or $V_2 = V(xy - 1)$. Specify in which case it is isomorphic to V_1 (resp. V_2). (Hint: Use linear changes of coordinates to eliminate monomials in F)